

Complete stationary surfaces in \mathbb{R}_1^4 with total curvature $-\int K dM = 4\pi$

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Abstract

Applying the general theory about complete spacelike stationary (i.e. zero mean curvature) surfaces in 4-dimensional Lorentz space \mathbb{R}_1^4 , we classify those regular algebraic ones with total Gaussian curvature $-\int K dM = 4\pi$. Such surfaces must be oriented and be congruent to either the generalized catenoids or the generalized enneper surfaces. For non-orientable stationary surfaces, we consider the Weierstrass representation on the oriented double covering \widetilde{M} (of genus g) and generalize Meeks and Oliveira's Möbius bands. The total Gaussian curvature are shown to be at least $2\pi(g+3)$ when $\widetilde{M} \rightarrow \mathbb{R}_1^4$ is algebraic-type. We conjecture that there do not exist non-algebraic examples with $-\int K dM = 4\pi$.

Keywords: stationary surface, Weierstrass representation, finite total Gaussian curvature, singular end, non-orientable surfaces

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1 Introduction

In a previous paper [5] we have generalized the classical theory about minimal surfaces in \mathbb{R}^3 to zero mean curvature spacelike surfaces in 4-dimensional Lorentz space. Such an immersed surface $M^2 \rightarrow \mathbb{R}_1^4$, called a *stationary surface* (see [1] for related works before), admits a Weierstrass-type representation formula, which involves a pair of meromorphic functions ϕ, ψ (the Gauss maps) and a holomorphic 1-form (the height differential) on M :

$$\mathbf{x} = 2 \operatorname{Re} \int \left(\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi \right) dh.$$

Among complete examples, those with finite total curvature are most important, i.e., when the integral of the Gaussian curvature $-\int K dM$ converges absolutely. For such surfaces, under mild assumptions we have established Gauss-Bonnet type formulas relating the total curvature with the Euler characteristic number of M , the generalized multiplicities \widetilde{d}_j of each ends, the mapping degree of ϕ, ψ , and the indices of the so-called *good singular ends*:

$$\int_M K dM = 2\pi \left(2 - 2g - r - \sum_{j=1}^r \widetilde{d}_j \right) = -2\pi \left(\deg \phi + \deg \psi - \sum_j |\operatorname{ind}_{p_j}| \right)$$

On this foundation, here we go on to consider complete examples with total curvature $-\int KdM = 4\pi$, which is the smallest possible value among algebraic stationary surfaces. (Here we ignore the trivial case when M is contained in a 3-dimensional degenerate subspace \mathbb{R}_0^3 . The induced metric is flat in that case with total curvature 0. See Section 2.)

Recall that in \mathbb{R}^3 , Osserman has shown that complete minimal surfaces with finite total curvature must be algebraic ones, i.e., they are given by meromorphic Weierstrass data over compact Riemann surfaces. In particular, immersed examples with $-\int KdM = 4\pi$ are either the catenoid or the Enneper surface. (For other complete minimal surfaces in \mathbb{R}^3 with small total curvature $-\int KdM \leq 12\pi$ and the classification results, see [2, 6].)

These two classical examples have been generalized by us in [5] to stationary surfaces in \mathbb{R}_1^4 (see Example 3.1 and 3.3). In this paper our main result is

Theorem A Let $x : M^2 \rightarrow \mathbb{R}_1^4$ be a complete, immersed, algebraic stationary surface with total curvature 4π . Then it is either a generalized catenoid, or a generalized Enneper surface. In particular, there does not exist non-orientable examples with $-\int KdM \leq 4\pi$.

Compared to minimal surfaces in \mathbb{R}^3 , here finite total Gaussian curvature (i.e., $\int_M KdM$ converges absolutely) still implies that M is conformally equivalent to a compact Riemann surface \overline{M} with finite punctures $\{p_j | 1 \leq j \leq r\}$. A main difference is that in our case, finite total curvature no longer implies *algebraic-type*. For counter-examples see Example 5.2 and Example 5.5. An interesting open problem is that whether there exist non-algebraic examples with $-\int KdM = 4\pi$. See discussions in Section 5.

Another new technical difficulty is that, to solve existence and uniqueness problems for complete stationary surfaces, now we must consider the following equation about complex variable z :

$$\phi(z) = \bar{\psi}(z), \tag{1}$$

We have to show that there are no solutions to it for meromorphic functions ϕ, ψ with given algebraic forms and certain parameters on a compact Riemann surface \overline{M} (except at several points assigned to be *good singular ends*). This is because that on an immersed surface there must be $\phi \neq \bar{\psi}$ (*regularity condition*). On the other hand, at one end where $\phi, \bar{\psi}$ take the same value with equal multiplicities (*bad singular end*), the total curvature will diverge. Such a complex equation (1) involving both holomorphic and anti-holomorphic functions is quite unusual to the knowledge of the authors. Most of the time we have to deal with this problem by handwork combined with experience. See [5] or Appendix A for related discussions. Note that $M \rightarrow \mathbb{R}^3$ is a rare case where we overcome this difficulty easily, because this time $\phi \equiv -1/\psi$, and this will never be equal to $\bar{\psi}$.

In [8], Meeks initiated the study of complete non-orientable minimal surfaces in \mathbb{R}^3 . Such surfaces are represented on its oriented double covering space, and the example with least possible total curvature 6π was constructed (Meeks' Möbius strip). Here we generalize this theory to non-orientable stationary surfaces in \mathbb{R}_1^4 (Section 4). A key result is the following lower bound estimation of the total curvature which helps to establish Theorem A above.

Theorem B Given a non-orientable surface M whose double covering space \widetilde{M}

has genus g and finite many ends, for any complete algebraic stationary immersion $x : M \rightarrow \mathbb{R}_1^4$ with finite total curvature there must be $-\int_M K dM \geq 2\pi(g+3)$.

We conjecture that $2\pi(g+3)$ is the best lower bound which could always be attained. Note that this agrees with the estimation for non-orientable minimal surfaces in \mathbb{R}^3 , and the conjecture is still open even in that special case [7].

We organize this paper as below. In Section 2 we review the basic theory about stationary surfaces in \mathbb{R}_1^4 . The orientable case and non-orientable case are discussed separately in Section 3 and 4. In Section 5 we give non-algebraic examples with small total curvature. The proofs to several technical lemmas are left to Appendix A and B.

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2 Preliminary

Let $\mathbf{x} : M^2 \rightarrow \mathbb{R}_1^4$ be an oriented complete spacelike surface in 4-dimensional Lorentz space. The Lorentz inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

We will briefly review the basic facts and global results established in [5] about such surfaces with zero mean curvature (called *stationary surfaces*).

Let $ds^2 = e^{2\omega}|dz|^2$ be the induced Riemannian metric on M with respect to a local complex coordinate $z = u + iv$. Hence

$$\langle \mathbf{x}_z, \mathbf{x}_z \rangle = 0, \quad \langle \mathbf{x}_z, \mathbf{x}_{\bar{z}} \rangle = \frac{1}{2}e^{2\omega}.$$

Choose null vectors \mathbf{y}, \mathbf{y}^* in the normal plane at each point such that

$$\langle \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{y}^*, \mathbf{y}^* \rangle = 0, \quad \langle \mathbf{y}, \mathbf{y}^* \rangle = 1, \quad \det\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{y}, \mathbf{y}^*\} > 0.$$

Such frames $\{\mathbf{y}, \mathbf{y}^*\}$ are determined up to scaling

$$\{\mathbf{y}, \mathbf{y}^*\} \rightarrow \{\lambda \mathbf{y}, \lambda^{-1} \mathbf{y}^*\} \tag{2}$$

for some non-zero real-valued function λ . After projection, we obtain two well-defined maps (independent to the scaling (2))

$$[\mathbf{y}], [\mathbf{y}^*] : M \rightarrow S^2 \cong \{[\mathbf{v}] \in \mathbb{R}P^3 | \langle \mathbf{v}, \mathbf{v} \rangle = 0\}.$$

The target space is usually called the projective light-cone, which is well-known to be homeomorphic to the 2-sphere. By analogy to \mathbb{R}^3 , we call them *Gauss maps* of the spacelike surface \mathbf{x} in \mathbb{R}_1^4 .

The surface has zero mean curvature $\vec{H} = 0$ if, and only if, $[\mathbf{y}], [\mathbf{y}^*] : M \rightarrow S^2$ are conformal mappings (yet they induce opposite orientations on S^2). Since

$S^2 \cong \mathbb{C} \cup \{\infty\}$, we may represent them locally by a pair of holomorphic and anti-holomorphic functions $\{\phi, \bar{\psi}\}$. The Weierstrass-type representation of stationary surface $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ is given by [5]:

$$\mathbf{x} = 2 \operatorname{Re} \int (\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi) dh \quad (3)$$

in terms of two meromorphic functions ϕ, ψ and a holomorphic 1-form dh locally. We call ϕ, ψ the Gauss maps of \mathbf{x} and dh the height differential.

Remark 2.1. When $\phi \equiv \mp 1/\psi$, by (3) we obtain a minimal surface in \mathbb{R}^3 , or a maximal surface in \mathbb{R}_1^3 . This recovers the Weierstrass representation in these classical cases. When ϕ or ψ is constant, we get a zero mean curvature spacelike surface in the 3-space $\mathbb{R}_0^3 \triangleq \{(x_1, x_2, x_3, x_3) \in \mathbb{R}_1^4\}$ with an induced degenerate inner product, which is essentially the graph of a harmonic function $x_3 = f(x_1, x_2)$ on complex plane $\mathbb{C} = \{x_1 + ix_2\}$.

Convention: In this paper, we always assume that neither of ϕ, ψ is a constant unless it is stated otherwise. According to the remark above, we have ruled out the trivial case of stationary surfaces in \mathbb{R}_0^3 . (According to (7) below, such surfaces have flat metrics and zero total Gaussian curvature.)

Remark 2.2. The induced action of a Lorentz orthogonal transformation of \mathbb{R}_1^4 on the projective light-cone is nothing but a Möbius transformation on S^2 , or equivalently, a fractional linear transformation on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. The Gauss maps ϕ, ψ and the height differential dh transform as below:

$$\phi \Rightarrow \frac{a\phi + b}{c\phi + d}, \quad \psi \Rightarrow \frac{\bar{a}\psi + \bar{b}}{\bar{c}\psi + \bar{d}}, \quad dh \Rightarrow (c\phi + d)(\bar{c}\psi + \bar{d})dh. \quad (4)$$

This is repeatedly used in Section 2 and Section 3 to simplify or to normalize the representation of examples.

Theorem 2.3. [5] *Given holomorphic 1-form dh and meromorphic functions $\phi, \psi : M \rightarrow \mathbb{C} \cup \{\infty\}$ globally defined on a Riemann surface M . Suppose they satisfy the regularity condition 1), 2) and period conditions 3) as below:*

- 1) $\phi \neq \bar{\psi}$ on M and their poles do not coincide;
- 2) The zeros of dh coincide with the poles of ϕ or ψ with the same order;
- 3) Along any closed path the periods satisfy

$$\oint_{\gamma} \phi dh = -\overline{\oint_{\gamma} \psi dh}, \quad (\text{horizontal period condition}) \quad (5)$$

$$\operatorname{Re} \oint_{\gamma} dh = \operatorname{Re} \oint_{\gamma} \phi \psi dh = 0. \quad (\text{vertical period condition}) \quad (6)$$

Then (3) defines a stationary surface $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$.

Conversely, any stationary surface $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ can be represented as (3) in terms of such ϕ, ψ and dh over a (necessarily non-compact) Riemann surface M .

The structure equations and the integrability conditions are given in [5]. An extremely important corollary is the formula below for the total Gaussian and normal curvature over a compact stationary surface M with boundary ∂M :

$$\begin{aligned} \int_M (-K + iK^\perp) dM &= 2i \int_M \frac{\phi_z \bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2} dz \wedge d\bar{z} \\ &= -2i \int_{\partial M} \frac{\phi_z}{\phi - \bar{\psi}} dz = -2i \int_{\partial M} \frac{\bar{\psi}_{\bar{z}}}{\phi - \bar{\psi}} d\bar{z}. \end{aligned} \quad (7)$$

At one end p with $\phi = \bar{\psi}$, the integral of total curvature above will become an improper integral. An important observation in [5] is that this improper integral converges absolutely only for a special class of such ends.

Definition 2.4. Suppose $\mathbf{x} : D - \{0\} \rightarrow \mathbb{R}_1^4$ is an annular end of a regular stationary surface (with boundary) whose Gauss maps ϕ and ψ extend to meromorphic functions on the unit disk $D \subset \mathbb{C}$. It is called a regular end when

$$\phi(0) \neq \bar{\psi}(0). \quad (\text{Thus } \phi(z) \neq \bar{\psi}(z), \forall z \in D.)$$

It is a singular end if $\phi(0) = \bar{\psi}(0)$ where the value could be finite or ∞ .

When the multiplicities of ϕ and $\bar{\psi}$ at $z = 0$ are equal, we call $z = 0$ a bad singular end. Otherwise it is a good singular end.

Proposition 2.5. [5] A singular end of a stationary surface $\mathbf{x} : D - \{0\} \rightarrow \mathbb{R}_1^4$ is good if and only if the curvature integral (7) converges absolutely around this end.

For a good singular end we introduced the following definition of its index.

Definition 2.6. [5] Suppose p is an isolated zero of $\phi - \bar{\psi}$ in p 's neighborhood D_p , where holomorphic functions ϕ and ψ take the value $\phi(p) = \bar{\psi}(p)$ with multiplicity m and n , respectively. The index of $\phi - \bar{\psi}$ at p (when ϕ, ψ are both holomorphic at p) is

$$\text{ind}_p(\phi - \bar{\psi}) \triangleq \frac{1}{2\pi i} \oint_{\partial D_p} d \ln(\phi - \bar{\psi}) = \begin{cases} m, & m < n; \\ -n, & m > n. \end{cases} \quad (8)$$

The absolute index of $\phi - \bar{\psi}$ at p is

$$\text{ind}_p^+(\phi - \bar{\psi}) \triangleq |\text{ind}_p(\phi - \bar{\psi})|. \quad (9)$$

For a regular end our index is still meaningful with $\text{ind} = \text{ind}^+ = 0$. For convenience we also introduce

$$\text{ind}^{1,0} \triangleq \frac{1}{2}(\text{ind}^+ + \text{ind}), \quad \text{ind}^{0,1} \triangleq \frac{1}{2}(\text{ind}^+ - \text{ind}), \quad (10)$$

which are always non-negative.

Note that our definition of index of $\phi - \bar{\psi}$ is invariant under the action of fractional linear transformation (4). So it is well-defined for a good singular end of a stationary surface. In particular, we can always assume that our singular ends do not coincide with poles of ϕ, ψ ; hence the definition above is valid.

A stationary surface in \mathbb{R}_1^4 is called an *algebraic stationary surface* if there exists a compact Riemann surface \bar{M} with $M = \bar{M} \setminus \{p_1, p_2, \dots, p_r\}$ such that $\mathbf{x}_z dz$ is a vector valued meromorphic form defined on \bar{M} . In other words, the Gauss map ϕ, ψ and height differential dh extend to meromorphic functions/forms on \bar{M} . For this surface class we have established Gauss-Bonnet type formulas involving the indices of the good singular ends.

Theorem 2.7 ([5]). For a complete algebraic stationary surface $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ given by (3) in terms of ϕ, ψ, dh without bad singular ends, the total Gaussian curvature

and total normal curvature are related with the indices at the ends p_j (singular or regular) by the following formulas:

$$\int_M K^\perp dM = 0, \quad (11)$$

$$\int_M K dM = -4\pi \left(\deg \phi - \sum_j \text{ind}^{1,0}(\phi - \bar{\psi}) \right) \quad (12)$$

$$= -4\pi \left(\deg \psi - \sum_j \text{ind}^{0,1}(\phi - \bar{\psi}) \right), \quad (13)$$

From (12)(13) we have equivalent identities:

$$\sum_j \text{ind}_{p_j}(\phi - \bar{\psi}) = \deg \phi - \deg \psi. \quad (14)$$

$$\int_M K dM = -2\pi \left(\deg \phi + \deg \psi - \sum_j \text{ind}_{p_j}^+(\phi - \bar{\psi}) \right). \quad (15)$$

Definition 2.8. The multiplicity of a regular or singular end p_j for a stationary surface in \mathbb{R}_1^4 is defined to be

$$\tilde{d}_j = d_j - \text{ind}_{p_j}^+,$$

where $d_j + 1$ is equal to the order of the pole of $\mathbf{x}_z dz$ at p_j .

Theorem 2.9 (Generalized Jorge-Meeks formula [5]). Given an algebraic stationary surface $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ with only regular or good singular ends $\{p_1, \dots, p_r\} = \bar{M} - M$. Let g be the genus of compact Riemann surface \bar{M} , r the number of ends, and \tilde{d}_j the multiplicity of p_j . We have

$$\int_M K dM = 2\pi \left(2 - 2g - r - \sum_{j=1}^r \tilde{d}_j \right), \quad \int_M K^\perp dM = 0. \quad (16)$$

Proposition 2.10 ([5]). Let $\mathbf{x} : D^2 - \{0\} \rightarrow \mathbb{R}_1^4$ be a regular or a good singular end which is further assumed to be complete at $z = 0$. Then its multiplicity satisfies $\tilde{d} \geq 1$.

Corollary 2.11 (The Chern-Osserman type inequality [5]). Let $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ be an algebraic stationary surface without bad singular ends, $\bar{M} = M \cup \{q_1, \dots, q_r\}$. Then

$$\int K dM \leq 2\pi(\chi(M) - r) = 4\pi(1 - g - r). \quad (17)$$

Corollary 2.12 (Quantization of total Gaussian curvature [5]). Under the same assumptions of the theorem above, when ϕ, ψ are not constants (equivalently, when \mathbf{x} is not a flat surface in \mathbb{R}_0^3), there is always

$$-\int_M K dM = 4\pi k \geq 4\pi,$$

where $k \geq 1$ is a positive integer.

3 Orientable case and examples with $-\int K dM = 4\pi$

This section is dedicated to the classification of complete stationary surfaces immersed in \mathbb{R}_1^4 with finite Gaussian curvature $-\int K dM = 4\pi$ which are orientable and of algebraic type.

Under our hypothesis, the generalized Jorge-Meeks formula (16) yields

$$r + \sum \tilde{d}_j + 2g = 4, \quad (18)$$

and the index formulas (12)(13) read

$$\deg \phi - \sum \text{ind}^{1,0} = 1, \quad \deg \psi - \sum \text{ind}^{0,1} = 1. \quad (19)$$

Since $r \geq 1$, and $\tilde{d}_j \geq 1$ for any end, there must be $g \leq 1$, and we need only to consider five cases separately as below.

- **Case 1:** $g = 1, r = 1, \tilde{d} = 1$ (torus with one end).

Since there is only one end, at least one of the indices $\text{ind}^{1,0}, \text{ind}^{0,1}$ is zero. By (19) we know either ϕ or ψ is a meromorphic function of degree 1. Yet this contradicts the well-known fact that over a torus there do not exist such functions. So we rule out this possibility.

- **Case 2:** $g = 0, r = 1, \tilde{d} = 3$ and the unique end is regular.

Such examples exist and they are generalization of the classical Enneper surface.

Example 3.1 (The generalized Enneper surfaces). *This is given by*

$$\phi = z, \quad \psi = \frac{c}{z}, \quad dh = s \cdot z dz, \quad (20)$$

or

$$\phi = z + 1, \quad \psi = \frac{c}{z}, \quad dh = s \cdot z dz, \quad (21)$$

over \mathbb{C} with complex parameters $c, s \in \mathbb{C} \setminus \{0\}$. \mathbf{x} has no singular points if and only if the parameter $c = c_1 + ic_2$ is not zero or positive real numbers in (20), or

$$c_1 - c_2^2 + \frac{1}{4} < 0 \quad (22)$$

in (21). When $c = -1$ in (20) we obtain the Enneper surface in \mathbb{R}^3 .

Indeed they are all examples in Case 2 according to the following result in [5].

Theorem 3.2. [5] *A complete immersed algebraic stationary surface in \mathbb{R}_1^4 with $\int K = -4\pi$ and one regular end is a generalized Enneper surface.*

- **Case 3:** $g = 0, r = 1, \tilde{d} = 3$ with a good singular end.

Suppose there exists such an example. Without loss of generality we assume that the singular end p has positive index. Since $\text{ind} \geq 1$, by definition we know that at p the function ψ takes the value $\psi(p)$ with multiplicity at least 2. On the other hand, $\text{ind}^{0,1} = 0$ and $\deg \psi = 1$, which is a contradiction to the observation above. Hence such examples do not exist.

- **Case 4:** $g = 0, r = 2, \tilde{d}_j = 1$ and both ends are regular.

The classical catenoid is one of such examples. The generalization in \mathbb{R}_1^4 is

Example 3.3 (The generalized catenoids). *This is defined over $M = \mathbb{C} \setminus \{0\}$ with*

$$\phi = z + t, \quad \psi = \frac{-1}{z-t}, \quad dh = s \frac{z-t}{z^2} dz. \quad (-1 < t < 1, s \in \mathbb{R} \setminus \{0\}) \quad (23)$$

When $t = 0$, it is the classical catenoid in \mathbb{R}^3 .

Theorem 3.4. [5] *A complete immersed algebraic stationary surface in \mathbb{R}_1^4 with total curvature $\int K = -4\pi$ and two regular ends is a generalized catenoid.*

• **Case 5:** $g = 0, r = 2, \tilde{d}_j = 1$ **with at least one good singular ends.**

This is the most difficult case in our discussion. We will show step by step that there are no such examples.

First, assume there is such a surface. We assert that it must have two singular ends whose indices have opposite signs. Otherwise, if there is only one good singular end which might be assumed to have positive index, similar to the discussion in Case 3 we can show $\deg \psi = 1$ and ψ has multiplicity greater than 1 at the end, which is a contradiction. In the same way we can rule out the possibility that both ends are singular with the same signs.

Second, without loss of generality we may suppose $M = \mathbb{C} \setminus \{0\}$ and the good singular ends are 0 and ∞ , with $\text{ind}_0 = m \geq 1, \text{ind}_\infty < 0$. By (19), $\text{ind}^{1,0} = m, \deg \phi = m + 1$. If $\text{ind}_\infty \leq -m - 1$, by definition we know ψ has multiplicity at least $m + 1$ at $z = \infty$ where ϕ must have higher multiplicity, which is impossible since $\deg \phi = m + 1$. If $\text{ind}_\infty \geq -m + 1$, by definition and (19) we know $\text{ind}^{0,1} \leq m - 1, \deg \psi \leq m$, which contradicts the requirement that ψ must have multiplicity greater than m at the first end $z = 0$. In summary there must be

$$\text{ind}_0 = m \geq 1, \quad \text{ind}_\infty = -m, \quad \deg \phi = \deg \psi = m + 1 \geq 2. \quad (24)$$

We observe that $\phi(0) \neq \phi(\infty)$. Otherwise, since $z = 0, \infty$ are both singular ends, there must be $\psi(0) = \phi(0) = \phi(\infty) = \psi(\infty)$. Because $z = 0$ is a good singular end and $\text{ind}_0 = m$, ψ has multiplicity at least $m + 1$ at $z = 0$ and multiplicity m at ∞ . This is impossible when $\deg \psi = m + 1, m \geq 1$.

This observation enables us to make the following normalization. Without loss of generality, suppose $\phi(0) = \psi(0) = 0, \phi(\infty) = \psi(\infty) = \infty$. Since meromorphic functions ϕ, ψ must be rational functions satisfying restrictions (24), we know

$$\phi(z) = z^m(z - a), \quad \psi(z) = \frac{z^{m+1}}{z - b}, \quad dh = \rho \frac{z - b}{z^k} dz, \quad (25)$$

where a, b, ρ are arbitrary nonzero complex parameters. Note that dh takes the form as above because M is regular at $z = b$. On the other hand, at the ends $z = 0$ and $z = \infty$ it should satisfy $\tilde{d}_0 \geq 1, \tilde{d}_\infty \geq 1$ according to Proposition 2.10, which implies $k = m + 2$ by the definition of \tilde{d} .

After fixing the form of ϕ, ψ, dh , we verify the period conditions. It is easy to see that the vertical period conditions are satisfied. The horizontal period conditions are satisfied if and only if $a + b = -\bar{\rho}/\rho$. In summary, such examples have Weierstrass data

$$\phi(z) = z^m(z - a), \quad \psi(z) = \frac{z^{m+1}}{z - b}, \quad dh = \rho \frac{z - b}{z^{m+2}} dz, \quad (26)$$

with parameters

$$m \geq 1, \quad a, b, \rho \in \mathbb{C} \setminus 0, \quad a + b = -\bar{\rho}/\rho. \quad (27)$$

If we can find nonzero parameters a, b, ρ as above so that the regularity condition $\phi \neq \bar{\psi}$ holds true for any $z \in \mathbb{C} \cup \{\infty\}$, then new examples with $-\int K dM = 4\pi$ are found. But according to Lemma 6.1 in Appendix A, for any given nonzero parameters a, b, ρ there always exist nonzero solutions z to the equation $\phi(z) = \bar{\psi}(z)$ for ϕ, ψ given above. We conclude that there exist no examples in Case 5, The proof to the following theorem has been finished.

Theorem 3.5. *Complete regular algebraic stationary surfaces $x : M \rightarrow \mathbb{R}_1^4$ with $-\int K dM = 4\pi$ are either the generalized catenoids or the generalized Enneper surfaces under the assumption that M is orientable.*

Another interesting observation is that if we make change of variables $z = w^2$ in (25), and choose the power k to be a even number suitably, then the period conditions always hold true and we don't need the restriction $a + b = -1$ in (27). In this situation, if parameters $a = b$ is chosen suitably, the regularity condition $\phi \neq \bar{\psi}$ is satisfied. See Lemma 6.3. In this way we find a complete, immersed stationary surface in \mathbb{R}_1^4 , yet with total curvature $-\int K dM = 8\pi$. See the example below (which has appeared in [5]).

Example 3.6 (Genus zero, two good singular ends and $\int_M K dM = -8\pi$).

$$M = \mathbb{C} \setminus \{0\}, \quad \phi = w^2(w^2 - a), \quad \psi = \frac{w^4}{w^2 - a}, \quad dh = \frac{w^2 - a}{w^4} dw. \quad (a \in \mathbb{C} \setminus \{0\})$$

The regularity, completeness and period conditions are satisfied when $-a$ is a sufficiently large positive real number (e.g. $-a > 1$). For the proof of regularity, see Lemma 6.3.

4 Non-orientable stationary surfaces and examples

In this section we will consider non-orientable algebraic stationary surfaces and show that the total curvature of them is always greater than 4π . For this purpose we need to consider their oriented double covering surface \widetilde{M} , and characterize the Weierstrass data over \widetilde{M} . This is a natural extension of Meeks' characterization of non-orientable minimal surfaces in \mathbb{R}^3 [8].

4.1 Representation of non-orientable stationary surfaces

Theorem 4.1. *Let \widetilde{M} be a Riemann surface with an anti-holomorphic involution $I : \widetilde{M} \rightarrow \widetilde{M}$ (i.e., a conformal automorphism of \widetilde{M} reversing the orientation) without fixed points. Let $\{\phi, \psi, dh\}$ be a set of Weierstrass data on \widetilde{M} such that*

$$\phi \circ I = \bar{\psi}, \quad \psi \circ I = \bar{\phi}, \quad I^* dh = \bar{dh}, \quad (28)$$

which satisfy the regularity and period conditions as well. Then they determine a non-orientable stationary surface

$$M = \widetilde{M} / \{\text{id}, I\} \rightarrow \mathbb{R}_1^4$$

by the Weierstrass representation formula (3).

Conversely, any non-orientable stationary surface $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ could be constructed in this way.

Remark 4.2. Geometrically, (28) is the consequence of reversing the orientation of the tangent plane by $z \rightarrow \bar{z}$, and reversing the induced orientation of the normal plane by interchanging the lightlike normal directions $[\mathbf{y}]$, $[\mathbf{y}^*]$.

Proof to Theorem 4.1. We prove the converse first. It is well-known that any non-orientable surface M has a orientable two-sheeted covering surface \widetilde{M} with an orientation-reversing homeomorphism I , and M is realized as the quotient surface

$$M = \widetilde{M}/\mathbb{Z}_2 = \widetilde{M}/\{\text{id}, I\}.$$

Denote π the quotient map. Notice that \widetilde{M} is endowed with the complex structure induced from the metric. When z is a local complex coordinate over a domain $U \subset \widetilde{M}$ which projects to M one-to-one, \bar{z} is also a coordinate over $I(U)$ compatible with the orientation on \widetilde{M} .

Consider the stationary surface $\tilde{\mathbf{x}} \triangleq \mathbf{x} \circ \pi : \widetilde{M} \rightarrow \mathbb{R}_1^4$. In the chart (U, z) we have

$$\tilde{\mathbf{x}}_z dz = (\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi) dh.$$

Then in the corresponding chart $(I(U), w = \bar{z})$, consider $\tilde{\mathbf{x}}^* = \tilde{\mathbf{x}} \circ I : I(U) \rightarrow \mathbb{R}_1^4$ and we have

$$\tilde{\mathbf{x}}_w^* dw = (\bar{\phi} + \bar{\psi}, i(\bar{\phi} - \bar{\psi}), 1 - \bar{\phi}\bar{\psi}, 1 + \bar{\phi}\bar{\psi}) d\bar{h}.$$

This implies (28).

Now we prove the first part. If $M = \widetilde{M}/\{\text{id}, I\}$ as described in the theorem and ϕ, ψ, dh satisfy condition (28) as well as the regularity and period conditions, then the integral along any path $\gamma \subset \widetilde{M}$ yields two stationary surfaces

$$\begin{aligned} \tilde{\mathbf{x}} &= 2 \operatorname{Re} \int_{\gamma} (\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi) dh, \\ \tilde{\mathbf{x}} \circ I &= \tilde{\mathbf{x}}^* = 2 \operatorname{Re} \int_{\gamma} (\bar{\psi} + \bar{\phi}, -i(\bar{\psi} - \bar{\phi}), 1 - \bar{\psi}\bar{\phi}, 1 + \bar{\psi}\bar{\phi}) d\bar{h}. \end{aligned}$$

If we assign the same initial value, then after either integration above we get the same result, because they are the real parts of a holomorphic vector-valued function and its complex conjugate. So $p \in \widetilde{M}$ and $I(p) \in \widetilde{M}$ are mapped to the same point in \mathbb{R}_1^4 , yet with opposite induced orientations on the same surface. After taking quotient we get a stationary immersion of the non-orientable M into \mathbb{R}_1^4 . This finishes the proof. \square

As an application of this theorem, we give a natural generalization of Meeks and Oliveira's construction of minimal Möbius strip.

Example 4.3 (Generalization in \mathbb{R}_1^4 of Meeks' minimal Möbius strip). *This is defined on $\widetilde{M} = \mathbb{C} \setminus \{0\}$ with involution $I : z \rightarrow -1/\bar{z}$, and the Weierstrass data be*

$$\phi = \frac{z - \lambda}{z - \bar{\lambda}} \cdot z^{2m}, \quad \psi = \frac{1 + \bar{\lambda}z}{1 + \lambda z} \cdot \frac{1}{z^{2m}}, \quad dh = i \frac{(z - \bar{\lambda})(1 + \lambda z)}{z^2} dz, \quad (29)$$

where λ is a complex parameter satisfying $\lambda \neq \pm 1$, $|\lambda| = 1$, and the integer $m \geq 1$.

Remark 4.4. When $\lambda = \pm i$ we have $\phi = -1/\psi$, and the example above is equivalent to Oliveira's examples in \mathbb{R}^3 [9]. (Meeks' example [8] corresponds to the case $m = 1$.) Otherwise this is a full map in \mathbb{R}_1^4 . Furthermore, for fixed m these examples are not

congruent to each other unless the values of the parameter λ are the same or differ by complex conjugation, because the cross ratio

$$\text{cr}(0, \infty; \lambda, \bar{\lambda}) = \frac{\lambda}{\bar{\lambda}}$$

between the zeros and poles in the normal form of ϕ is an invariant.

Proposition 4.5. *Example 4.3 is a complete immersed stationary Möbius strip with a regular end and total Gaussian curvature $2(2m+1)\pi$.*

Proof. We start from a general case, a Möbius strip $M = \widetilde{M}/\{\text{id}, I\} \rightarrow \mathbb{R}_1^4$ with

$$\widetilde{M} = \mathbb{C} \setminus \{0\}, \quad I : z \rightarrow -1/\bar{z}, \quad \phi(z) = \frac{az+b}{cz+d} \cdot z^{2m}. \quad (a, b, c, d \in \mathbb{C}, ad - bc \neq 0)$$

To satisfy condition (28), there should be

$$\psi = \overline{\phi(-1/\bar{z})} = \frac{\bar{b}z - \bar{a}}{\bar{d}z - \bar{c}} \cdot \frac{1}{z^{2m}}.$$

The surface is regular outside the ends $\{0, \infty\}$. Together with $dh^* = \overline{dh}$, this implies

$$dh = i \frac{(cz+d)(\bar{d}z - \bar{c})}{z^2} dz$$

up to multiplication by a real constant. Under these conditions it is easy to verify that the metric is complete.

Next, let us check the period conditions. The horizontal periods vanish automatically since $\phi dh, \psi dh$ has no residues at 0 and ∞ . The vertical periods must vanish, hence $|d|^2 = |c|^2$, $|b|^2 = |a|^2$. Without loss of generality we may write

$$\phi = \frac{z - \lambda}{z - \bar{\lambda}} \cdot z^{2m}, \quad |\lambda| = 1.$$

To simplify ϕ to this form we have utilized the freedom to change complex coordinate by $z \rightarrow \mu z$ and the (fractional) linear transformation $\phi \rightarrow \mu' \phi$ induced from the Lorentz transformation of \mathbb{R}_1^4 (see (4)).

We are left to verify $\phi \neq \bar{\psi}$ over $\mathbb{C} \setminus \{0\}$. (At the ends $z = 0, \infty$ it is obviously true. So they are regular ends.) Suppose $\phi(z) = \bar{\psi}(z)$ for some $z \in \mathbb{C}$. Substitute the expressions of ϕ, ψ into it. We obtain

$$|z|^{4m} = \frac{(z - \bar{\lambda})(-1/\bar{z} - \lambda)}{(z - \lambda)(-1/\bar{z} - \bar{\lambda})} = \text{cr}(z, -1/\bar{z}; \bar{\lambda}, \lambda).$$

Since the cross ratio at the right hand side takes a real value, four points $z, \frac{-1}{\bar{z}}, \bar{\lambda}, \lambda$ are located on a circle C in the complex plane \mathbb{C} .

We assert that this circle C could not be identical to the unit circle. (Otherwise $|z| = 1$ and the cross ratio above is 1. This holds true only if $z = \frac{-1}{\bar{z}}$, which is impossible, or $\lambda = \bar{\lambda} = \pm 1$, which has been ruled out in Example 4.3.)

Circle C intersects the unit circle at λ and $\bar{\lambda}$. Observe that any circle passing through $z, \frac{-1}{\bar{z}}$ will intersect the unit circle at an antipodal point pair. (Because under the inverse of the standard stereographic projection, $z, \frac{-1}{\bar{z}}$ correspond to two antipodal points on S^2 , and the unit circle corresponds to the equator. Any circle passing through the inverse images of $z, \frac{-1}{\bar{z}}$ on S^2 will intersect the equator again at two antipodal points. After taking stereographic projection back to \mathbb{C} we get the conclusion.) As a consequence, $\lambda = \pm i$. But this time the aforementioned cross ratio could only take value as a negative real number (because on circle C , $z, \frac{-1}{\bar{z}}$ must be separated by $\pm i$). This contradiction finishes our proof. \square \square

When $m = 1$ this example has smallest possible total curvature 6π among non-orientable algebraic stationary surfaces. (Note that the classical Henneberg surface in \mathbb{R}^3 has total curvature 2π , yet with four branch points.) This conclusion is the corollary of a series of propositions below.

4.2 Non-orientable stationary surfaces of least total curvature

In general we are interested in finding least possible total curvature for non-orientable stationary surfaces of a given topological type. This is motivated by discussions of F. Martin in [7]. Compared with minimal surfaces in \mathbb{R}^3 , this general case looks even more interesting (at least to the authors).

As a consequence of Theorem 4.1, for a complete non-orientable stationary surface with double covering \widetilde{M} of genus g with $2r$ ends, there must be $\deg \phi = \deg \psi$; the index formula (15) as well as the Jorge-Meeks formula (16) implies

$$-\int_M K = 2\pi \left(\deg \phi - \sum_{j=1}^r |\text{ind}_{p_j}| \right) = 2\pi \left(g + r - 1 + \sum_{j=1}^r \tilde{d}_j \right). \quad (30)$$

Because $r \geq 1$ and $\tilde{d}_j \geq 1$, we know

$$-\int_M K \geq 2\pi(g + 1).$$

A better estimation is given in the following proposition.

Proposition 4.6. *Given a non-orientable surface M whose double covering space \widetilde{M} has genus g and finite many punctures, there does not exist complete algebraic stationary immersion $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ with total Gaussian curvature $-\int_M K dM = 2\pi(g + 1)$. In other words, under our assumptions there must be*

$$-\int_M K dM \geq 2\pi(g + 2). \quad (31)$$

Proof. Consider the lift of \mathbf{x} , i.e., $\tilde{\mathbf{x}} : \widetilde{M} \rightarrow \mathbb{R}_1^4$. Since the immersion is algebraic and $-\int_{\widetilde{M}} K < +\infty$, it has finite many regular or good singular ends, and the total number is a even number $2r$ (r is the number of ends of M). By the modified Jorge-Meeks formula (30) and $r \geq 1$, a Chern-Osserman type inequality is obtained:

$$-\int_M K dM \geq 2\pi(g + 1).$$

Suppose the equality is achieved. Then there must be two ends for \widetilde{M} and $\tilde{d}_1 = \tilde{d}_2 = 1$. Both of them are regular ends or good singular ends at the same time. We will show that in either case there will be a contradiction.

Case 1: regular end(s). The multiplicity $\tilde{d}_1 = d_1 = 1$, and $\tilde{\mathbf{x}}_z dz$ for the end p_1 has a pole of order 2. In a local coordinate chart with $z(p_1) = 0$ we write out the Laurent expansion of $\tilde{\mathbf{x}}_z$:

$$\tilde{\mathbf{x}}_z = \frac{1}{z^2} \mathbf{v}_2 + \frac{1}{z} \mathbf{v}_1 + (\text{holomorphic part}).$$

Since this is a regular end, \mathbf{v}_2 is an isotropic vector whose real and imaginary parts span a 2-dimensional spacelike subspace. \mathbf{v}_1 is a real vector orthogonal to \mathbf{v}_2 by the

period condition and $\langle \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_z \rangle = 0$. Thus in \mathbb{R}_1^4 there exist a constant non-zero real vector $\mathbf{v}_0 \perp \mathbf{v}_2, \mathbf{v}_1$.

At the other end $p_2 = I(p_1)$ with local coordinate $w = \bar{z}$, because $\tilde{\mathbf{x}}_w = \tilde{\mathbf{x}}_{\bar{z}}$, we know the same \mathbf{v}_0 is orthogonal to the principal part of the Laurent series. Thus $\langle \tilde{\mathbf{x}}_z dz, \mathbf{v}_0 \rangle$ is a holomorphic 1-form, and $\langle \tilde{\mathbf{x}}, \mathbf{v}_0 \rangle$ is a harmonic function defined on the whole compact Riemann surface. It must be a constant; hence $\tilde{\mathbf{x}}$ as well as \mathbf{x} is contained in a 3-dimensional subspace of \mathbb{R}_1^4 .

Yet this is impossible. Since in \mathbb{R}_1^3 or \mathbb{R}_0^3 there exist no immersed spacelike non-oriented surfaces. The possibility of $M \subset \mathbb{R}^3$ could be ruled out by Schoen's famous result [11] that any complete, connected, oriented minimal surface in \mathbb{R}^3 with two embedded ends is congruent to the catenoid. (Alternatively, we may argue by the maximal principle once again. Since the unique end of M is an embedded end in \mathbb{R}^3 , which is either a catenoid end or a planar end, one can choose the coordinate of \mathbb{R}^3 suitably so that the height function \mathbf{x}_3 is bounded from below over the whole M . Such a harmonic function must be a constant, and $M \subset \mathbb{R}^2$. Contradiction.)

Case 2: good singular end(s). At the good singular end p_1 , without loss of generality, suppose it has $\text{ind} = m \geq 1$ and $\phi(p_1) = \psi(p_1) = 0$. Then $\tilde{d}_1 = d_1 - m = 1$, and $\tilde{\mathbf{x}}_z dz$ has a pole of order $m + 2$ at p_1 . There always exists a suitable local coordinate z such that $z(p_1) = 0$ and

$$dh = \frac{dz}{z^{m+2}}, \quad \phi(z) = a_0 z^m + a_1 z^{m+1} + O(z^{m+2}), \quad \psi(z) = b_1 z^{m+1} + O(z^{m+2}).$$

By (3) we know

$$\begin{aligned} \tilde{\mathbf{x}}_z dz &= (\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi) dh \\ &= \frac{dz}{z^{m+2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \frac{dz}{z^2} \begin{pmatrix} a_0 \\ -ia_0 \\ 0 \\ 0 \end{pmatrix} + \frac{dz}{z} \begin{pmatrix} a_1 + b_1 \\ -i(a_1 - b_1) \\ 0 \\ 0 \end{pmatrix} + (\text{holomorphic part}). \end{aligned}$$

Take $\mathbf{v}_0 = (0, 0, 1, 1)$. We can argue as in case 1 to show that $\langle \tilde{\mathbf{x}}, \mathbf{v}_0 \rangle$ is a harmonic function defined on the whole compact Riemann surface, hence be a constant. (The key point is that M has only one end.) Thus $\tilde{\mathbf{x}}(\widetilde{M})$ as well as $x(M)$ is contained in an affine space \mathbb{R}_0^3 (orthogonal to \mathbf{v}_0). Yet this is also impossible for a non-oriented spacelike surface. \square \square

We will show that the lower bound could be improved to be $2\pi(g + 3)$, the same as the case for non-orientable minimal surfaces in \mathbb{R}^3 .

Theorem 4.7. *Given a non-orientable surface M whose double covering space \widetilde{M} has genus g and finite many punctures, there does not exist complete algebraic stationary immersion $\mathbf{x} : M \rightarrow \mathbb{R}_1^4$ with total Gaussian curvature $-\int_M K dM = 2\pi(g + 2)$. In other words, under our assumptions there must be*

$$-\int_M K dM \geq 2\pi(g + 3). \quad (32)$$

Proof. As in the proof to Proposition 4.6, consider the lift of \mathbf{x} , i.e., $\tilde{\mathbf{x}} : \widetilde{M} \rightarrow \mathbb{R}_1^4$. Suppose the lower bound $2\pi(g + 2)$ is attained. Then \widetilde{M} has two ends p_1, p_2 with $\tilde{d}_1 = \tilde{d}_2 = 2$ by (30). By symmetry, both of them are regular or good singular ends at the same time. Each possibility is ruled out using different arguments.

When both ends are good singular ends, we use the same argument as in Case 2 of Proposition 4.6. At the good singular end p_1 , without loss of generality, suppose it has $\text{ind} = m \geq 1$ and $\phi(p_1) = \psi(p_1) = 0$. Then $\tilde{d}_1 = d_1 - m = 2$, and $\tilde{\mathbf{x}}_z dz$ has a pole of order $m + 3$ at p_1 . There always exists a suitable local coordinate z such that $z(p_1) = 0$ and

$$dh = \frac{dz}{z^{m+3}}, \quad \phi(z) = a_0 z^m + a_1 z^{m+1} + O(z^{m+2}), \quad \psi(z) = b_1 z^{m+1} + O(z^{m+2}).$$

By the Weierstrass representation formula we know

$$\begin{aligned} \tilde{\mathbf{x}}_z = & \frac{1}{z^{m+3}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{z^3} \begin{pmatrix} a_0 \\ -ia_0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{z^2} \begin{pmatrix} a_1 + b_1 \\ -i(a_1 - b_1) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} a_2 + b_2 \\ -i(a_2 - b_2) \\ a_0 b_1 z^{m-1} \\ -a_0 b_1 z^{m-1} \end{pmatrix} \\ & + (\text{holomorphic part}). \end{aligned}$$

Take $\mathbf{v}_0 = (0, 0, 1, 1)$. Because $m \geq 1$, $\langle \tilde{\mathbf{x}}, \mathbf{v}_0 \rangle$ is a harmonic function (with the leading term $\ln |z|$) bounded from below or above in a neighborhood of p_1 . Since M has only one end around which the assertion above is still valid, we conclude that $\langle \tilde{\mathbf{x}}, \mathbf{v}_0 \rangle$ is a harmonic function bounded from below or above over the whole compactified surface. It must be a constant, and the surface is contained in a 3-space. As in Proposition 4.6 this leads to a contradiction.

In case that both ends are regular, consider the anti-holomorphic automorphism $I : \tilde{M} \rightarrow \tilde{M}$ without fixed points and $M \cong \tilde{M}/\{I\}$. Under the assumptions above, the Gauss map ϕ could be viewed as a continuous map from the oriented double covering space to the round 2-sphere such that

$$\phi : \{\tilde{M}; I\} \rightarrow S^2 \subset \mathbb{R}^3, \quad \text{s.t.}, \quad \phi(p) \neq \phi(I(p)).$$

It is a standard fact that such a map is homotopic to an odd map

$$\tilde{\phi} \triangleq \frac{\phi(p) - \phi(I(p))}{|\phi(p) - \phi(I(p))|}, \quad \tilde{\phi}(I(p)) = -\tilde{\phi}(p),$$

where we give the homotopy $H(p, t) = \frac{\phi(p) - t\phi(I(p))}{|\phi(p) - t\phi(I(p))|}$ directly. According to Theorem 7.1 in Appendix B, the mapping degree of such an odd map and $g - 1$ must be both even or both odd. Thus the mapping degree could not be $g + 2$. This finishes the proof. \square \square

Conjecture 4.8. *The lower bound (32) $-\int_M K dM \geq 2\pi(g + 3)$ is sharp for any given $g \geq 0$. In other words, there always exists an complete, immersed, algebraic non-orientable stationary surfaces whose double covering surface has genus g .*

This is a generalization of conjecture 1 in [7] for non-orientable minimal surfaces in \mathbb{R}^3 . It is verified in \mathbb{R}^3 when $g = 0$ and $g = 1$. The corresponding examples are the Meeks' Möbius strip [8] and Lopez's Klein bottle [6]. For higher genus g this conjecture is still open.

As the direct consequence of Theorem 4.7 we obtain the following result:

Theorem 4.9. *There does not exist a complete, algebraic, immersed non-orientable stationary surface in \mathbb{R}_1^4 with total Gaussian curvature $-\int_M K dM = 4\pi$.*

Combined with Theorem 3.5, this finishes the proof to our classification theorem (Theorem A in the Introduction).

Remark 4.10. We note a significant difference between non-orientable stationary surfaces in \mathbb{R}_1^4 and \mathbb{R}^4 . Oliveira [9] constructed complete Möbius band in \mathbb{R}^4 with total curvature $2\pi m$ for any $m \geq 2$. So the total curvature 4π could be realized in that case.

In the proof to Theorem 4.7, when treating the special case with only regular ends, indeed we have obtained the following proposition, which is a partial generalization of Meeks' result (Corollary 1 in [8]):

Proposition 4.11. *A complete non-orientable stationary surface in \mathbb{R}_1^4 of algebraic type without singular ends must have total curvature $-\int_M K dM = 2\pi m$, where $m \equiv g - 1 \pmod{2}$, and g is the genus of the oriented double covering surface.*

So far we do not know whether it is true in the general case when good singular ends exist.

5 Non-algebraic examples with small total Gaussian curvature

Recall the following classical result.

Theorem 5.1. *Let (M, ds^2) be a non-compact surface with a complete metric. Suppose $\int_M |K| dM < +\infty$, then:*

(1) (Huber[3]) *There is a compact Riemann surface \overline{M} such that M as a Riemann surface is biholomorphic to $\overline{M} \setminus \{p_1, p_2, \dots, p_r\}$.*

(2) (Osserman[10]) *When this is a minimal surface in \mathbb{R}^3 with the induced metric ds^2 , the Gauss map $G = \phi = -1/\psi$ and the height differential dh extend to each end p_j analytically.*

(3) (Jorge and Meeks [4]) *As in (1) and (2), suppose minimal surface $M \rightarrow \mathbb{R}^3$ has r ends and \overline{M} is the compactification with genus g . The total curvature is related with these topological invariants via the Jorge-Meeks formula:*

$$\int_M K dM = 2\pi \left(2 - 2g - r - \sum_{j=1}^r d_j \right), \quad (33)$$

Here $d_j + 1$ equals to the highest order of the pole of $\mathbf{x}_z dz$ at p_j , and d_j is called the multiplicity at the end p_j .

Huber's conclusion (1) means *finite total curvature \Rightarrow finite topology*, which is a purely intrinsic result. In particular, this is valid also for stationary surfaces in \mathbb{R}_1^4 . Surprisingly, as to the extrinsic geometry, Osserman's result 2) is no longer true in \mathbb{R}_1^4 . In particular we have non-algebraic counter-examples given below:

Example 5.2 ($M_{k,a}$ with essential singularities and finite total curvature [5]).

$$M_{k,a} \cong \mathbb{C} - \{0\}, \quad \phi = z^k e^{az}, \quad \psi = -\frac{e^{az}}{z^k}, \quad dh = e^{-az} dz. \quad (34)$$

where integer k and real number a satisfy $k \geq 2, 0 < a < \frac{\pi}{2}$.

Proposition 5.3. [5] *Stationary surfaces $M_{k,a}$ in Examples 5.2 are regular, complete stationary surfaces with two ends at $z = 0, \infty$ satisfying the period conditions. Moreover their total curvature converges absolutely with*

$$\int_M K dM = -4\pi k, \quad \int_M K^\perp dM = 0. \quad (35)$$

Remark 5.4. Taking different height differential dh in Example 5.2, we can obtain other examples with the same total Gaussian curvature. Yet the total Gaussian curvature $-\int_M K dM = 4\pi$ could not be realized since when $k = 1$ the integral is not absolutely convergent.

Similar to the construction of Example 5.2 and the proof to Proposition 5.3 as in [5], we have non-oriented, non-algebraic examples as below.

Example 5.5 (stationary Möbius strips with essential singularities and finite total curvature).

$$\phi = z^{2p-1} e^{\frac{1}{2}(z-\frac{1}{z})}, \quad \psi = \frac{-1}{z^{2p-1}} e^{\frac{1}{2}(z-\frac{1}{z})}, \quad dh = d e^{-\frac{1}{2}(z-\frac{1}{z})}. \quad (p \in \mathbb{Z}_{\geq 2}) \quad (36)$$

Proposition 5.6. *Example 5.5 is a complete immersed stationary Möbius strip with finite total curvature $\int | -K + iK^\perp | dM < +\infty$. We have*

$$-\int_M K dM = 2(2p-1)\pi, \quad \int_M K^\perp dM = 0.$$

In particular, the smallest possible value of their total Gaussian curvature is 6π .

Proof. It is easy to verify $\phi^* = \bar{\psi}, \psi^* = \bar{\phi}, dh^* = \overline{dh}$. So we obtain a Möbius strip according to Theorem 4.1.

The regularity is also easy to verify. For example, if there exist z such that $\phi(z) = \bar{\psi}(z)$, by (36) we get $z\bar{z} e^{i \cdot \text{Im}(z-\frac{1}{z})} = -1$. So $|z| = 1$ and $z = e^{i\theta}$. Insert this back to the previous equation; we obtain $e^{2i \sin \theta} = -1$, which is impossible.

Next we check the period condition. Since

$$dh = d e^{-\frac{1}{2}(z-\frac{1}{z})}, \quad \phi\psi dh = d e^{\frac{1}{2}(z-\frac{1}{z})},$$

both being exact 1-forms, there are no vertical periods. At the same time,

$$\phi dh = -\frac{1}{2} \left(1 + \frac{1}{z^2}\right) z^{2p-1} dz, \quad \psi dh = -\frac{1}{2} \left(1 + \frac{1}{z^2}\right) \frac{-1}{z^{2p-1}} dz.$$

When $p \geq 2$ neither of these 1-forms has residue. So there are no horizontal periods.

By direct computation one can show that the integral of the absolute total curvature $\int | -K + iK^\perp | dM$ is asymptotic to $\int |z|^{2-4p} dz d\bar{z}$ when $z \rightarrow \infty$, or to $\int |z|^{4p-6} dz d\bar{z}$ when $z \rightarrow 0$. Thus when $p \geq 2$ the total curvature integral converges absolutely. Approximate \widehat{M} by domains $A_{r,R} \triangleq \{0 < r \leq |z| \leq R\}$. By Stokes theorem we get

$$\begin{aligned} \int_{A_{r,R}} (-K + iK^\perp) dM &= 2i \int_{A_{r,R}} \frac{\phi_z \bar{\psi}}{(\phi - \bar{\psi})^2} dz \wedge d\bar{z} \\ &= -2i \oint_{|z|=R} \frac{\phi_z}{\phi - \bar{\psi}} dz + 2i \oint_{|z|=r} \frac{\phi_z}{\phi - \bar{\psi}} dz, \end{aligned}$$

where

$$\frac{\phi_z}{\phi - \bar{\psi}} = \frac{|z|^{4p-2} \left(\frac{1}{2} + \frac{2p-1}{z} + \frac{1}{2z^2} \right)}{|z|^{4p-2} + e^{-i \cdot \text{Im}(z-\frac{1}{z})}}.$$

When $R \rightarrow \infty$ the first contour integral converges to $-2i(2p-1) \cdot 2\pi i$. When $r \rightarrow 0$ the second contour integral converges to 0. This completes the proof. $\square \quad \square$

In the discussion above, when $p = 1$ the horizontal period condition is violated, and the total curvature integral does not converge absolutely. Thus among these simplest examples (including Examples 5.2) we can not find one with total Gaussian curvature 4π . This motivates the following

Conjecture 5.7. *There does NOT exist any complete, non-algebraic stationary surfaces immersed in \mathbb{R}_1^4 with finite total Gaussian curvature $-\int K dM = 4\pi$.*

6 Appendix A

In the proof to our main theorem, a key fact is that a complete, algebraic stationary surface $M \cong \mathbb{C} \setminus \{0\}$ with two good singular ends and $-\int K dM = 4\pi$ must have other singular (branch) points. This follows from

Lemma 6.1. *For any positive integer $m \in \mathbb{Z}^+$ and any non-zero complex parameters $a, b \in \mathbb{C} \setminus \{0\}$ whose sum $a + b = -e^{it}$ is a given unit complex number ($t \in \mathbb{R}$), there always exists a solution $z \neq 0$ to the equation*

$$(\bar{z} - \bar{a})(z - b) = \frac{z^{m+1}}{\bar{z}^m}. \quad (37)$$

Proof. By change of coordinates $z \rightarrow ze^{it}$, we may consider an equivalent equation

$$(\bar{z} - \bar{a})(z - b) = \lambda \frac{z^{m+1}}{\bar{z}^m}, \quad (38)$$

where $\lambda = e^{it(2m+1)}$ is a unit complex number, and a, b are nonzero complex parameters satisfying

$$a + b = -1.$$

In other words, the middle point of the segment \overline{ab} is $-1/2$. We will show that there always exists a solution $z \neq 0$ to the equation (38) for any given unit complex number λ and when $a + b = -1$.

First let us explain the basic idea of our proof. Consider the equal-module locus

$$\Gamma = \{z \in \mathbb{C} : |z - a||z - b| = |z|\}$$

where the two sides of (38) have equal modules. Since $|z - a||z - b| < |z|$ when $z = a, b$, and $|z - a| \cdot |z - b| > |z|$ when z is big enough, by continuity we know this locus is non-empty. It is easy to see that $\Gamma = \cup \Gamma_j$ is a union of several continuous, connected, (simple) closed curves. Next we compare the argument of the complex functions at both sides of (38). Define

$$\arg_L \triangleq \arg[(\bar{z} - \bar{a})(z - b)], \quad \arg_R \triangleq \arg(z^{m+1}/\bar{z}^m) = (2m + 1) \arg(z).$$

Note that the two arguments \arg_L, \arg_R can be defined and extended continuously along any continuous path (without self-intersection) on \mathbb{C} . We want to find one component $\Gamma_j \subset \Gamma$ such that

$$\delta = \arg_R - \arg_L,$$

the difference of the two arguments, will have a bounded variation greater than 2π . Again by continuity we know that along Γ_j there is some point z at which both sides of (38) share equal modules and arguments. This will finish our proof.

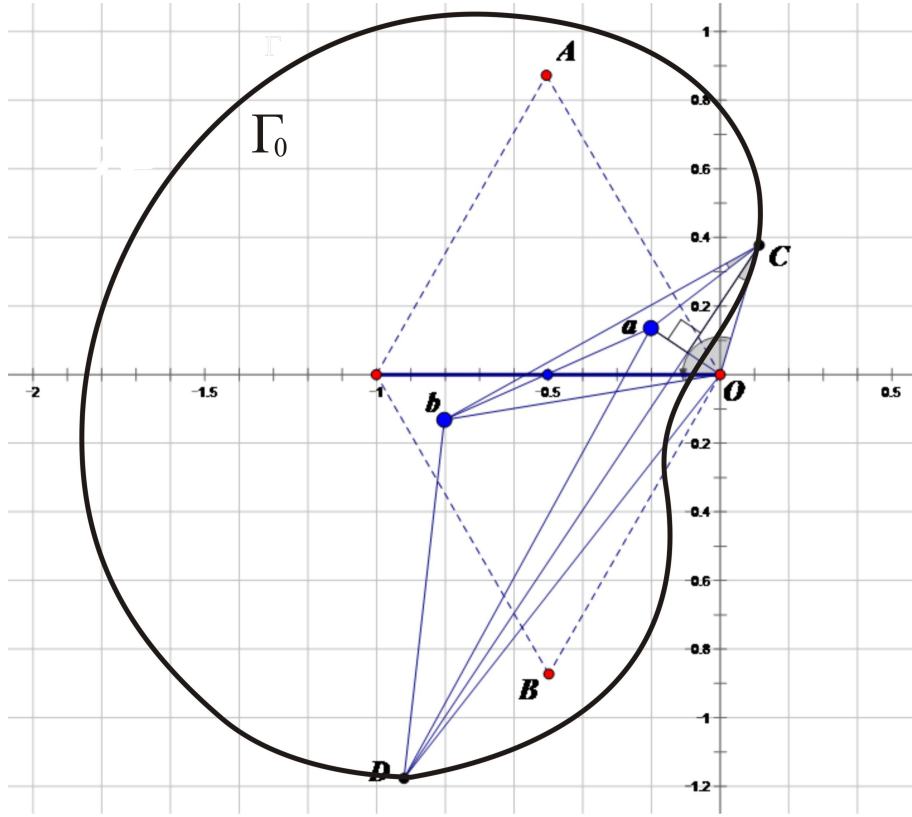


Figure 1: Case 1, $|a - b| \leq 1$

To estimate the variation of the argument difference δ , the next key point is to construct two points on the locus Γ using some elementary geometry. Now we have to consider two cases separately.

Case 1: $|a - b| \leq 1$.

In this case, the complex numbers a, b correspond to two points located inside the circle $|z + \frac{1}{2}| = \frac{1}{2}$ and being symmetric about $z = -\frac{1}{2}$.

Suppose $\text{Im}(a) \geq 0, \text{Im}(b) \leq 0$. We want to find two points C, D on Γ which are equidistant to a and the origin O , at the same time whose distance to b is 1 (see Figure 1). Such points are exactly the intersection between the bisector of the segment \overline{aO} and the unit circle centered at b . Because the length $|\overline{ba}|$ and $|\overline{bO}|$ are no more than 1 ($|b| \leq |b + 1/2| + |-1/2| \leq 1$), we know the intersection points C, D exist, and they are distinct. Let C be the one on the upper half plane.

We claim that C, D must be located on one and the same component (a simple closed curve) $\Gamma_0 \subset \Gamma$. Notice that the open segment \overline{CD} (on the bisector) is contained in the interior of the unit circle centered at b , hence also in the interior of

$$\Omega = \{z \in \mathbb{C} : |z - a||z - b| < |z|\}.$$

Let $\Gamma_0 \subset \Gamma = \partial\Omega$ be the component passing through C . Then the straight line CD must have at least one more intersection with Γ_0 , whose coordinate z satisfies $|z - a||z - b| = |z|$ and $|z - a| = |z|$, hence $|z - b| = 1$. It has to be D as defined above. This verifies our claim.

The main consequence of the condition $|a - b| \leq 1$ is that $|b| \leq 1$. Using the

relation that greater angle is opposite greater side in the triangle $\triangle bCO$, we know

$$2\angle OCD + \angle aCb = \angle OCb \leq \angle COb.$$

Similarly, in the triangle $\triangle bDO$ we have

$$2\angle ODC + \angle aDb = \angle ODb \leq \angle DOb.$$

Taking sum of these two equalities and using $\angle COb + \angle DOb = \pi - \angle OCD - \angle ODC$ in the triangle $\triangle OCD$, we get

$$\angle aCb + \angle aDb \leq 3(\angle COb + \angle DOb) - 2\pi \leq (2m+1)(\angle COb + \angle DOb) - 2\pi.$$

Notice that

$$\begin{aligned}\angle aCb &= -\arg_L(C) = [\arg(z-a) - \arg(z-b)]|_{z=C}, \\ \angle aDb &= \arg_L(D) = [\arg(z-b) - \arg(z-a)]|_{z=D}, \\ (2m+1)(\angle COb + \angle DOb) &= \arg_R(D) - \arg_R(C).\end{aligned}$$

Then the previous inequality amounts to say

$$\delta(D) - \delta(C) \geq 2\pi.$$

Thus along the continuous path connecting C, D which is part of the equal-module locus $\Gamma_0 \subset \Gamma$, the quotient between $(\bar{z} - \bar{a})(z - b)$ and z^{m+1}/\bar{z}^m can take any given unit complex parameter λ . This finishes the proof in the first case.

We observe that if $\text{Im}(a) \leq 0, \text{Im}(b) \geq 0$ (or just interchange a, b in Figure 1 above), the proof is similar.

It seems that our proof relies on the special case of Figure 1 where a is inside the triangle bCO . Indeed, because the positivity of these two angles was never used in that proof, when a is outside the triangle bCO the proof is still valid.

Case 2: $|a - b| > 1$.

Other than case 1, now we have to find a different way to construct such two points C, D on the equal-module locus.

Consider the triangle $\triangle Oab$. The length of the median on the side \overline{ab} is less than half of $|\overline{ab}|$. So $\angle aOb > \pi/2$. Moreover, any point C on the line segment Oa or Ob will span an obtuse angle $\angle aCb$. This is the main consequence of $|a - b| > 1$.

Let C be a moving point on the line segment Oa with coordinate z . Since $|z - a||z - b| < |z|$ when z is very close to a , and $|z - a||z - b| > |z|$ when z is very close to 0, there exist at least one intersection between \overline{Oa} and the equal-module locus Γ . We take C_1 to be the one closest to a among all such intersection points. Similarly, we take C_2 to be the one closest to b among all intersection points between \overline{Ob} and Γ .

On the straight line ab we can also find two intersection points with Γ , denoted as D_1, D_2 , such that D_1, b, a, D_2 are located on the line ab in the usual linear order, and D_1 (D_2) is the closest one among all intersection points between Γ and the ray aD_1 (aD_2).

Assume that a is on the upper half plane and b is on the lower half plane. (The other possibilities will be treated later.) We consider two subcases.

The first subcase is that C_1, D_1 are on the same connected component Γ_0 of Γ . Let us start from C_1 and end up with D_1 while turning counter-clockwise around a

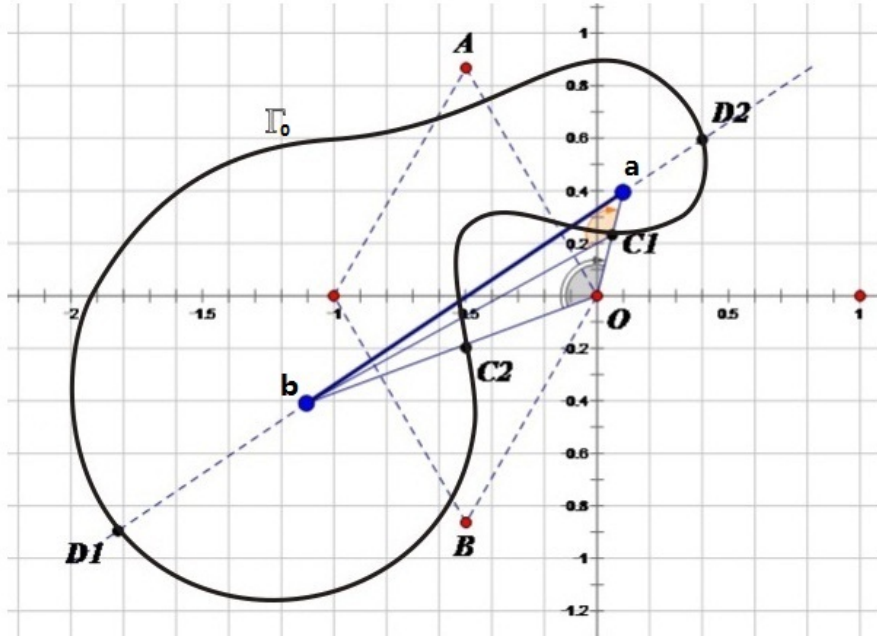


Figure 2: Subcase 2.1

along Γ_0 . Then it is easy to see that the variation of δ will be more than 2π because the increase of \arg_L is $\angle aC_1b > \frac{\pi}{2}$ and the decrease of \arg_R is $(2m+1)\angle D_1OC_1 > \frac{(2m+1)\pi}{2}$. As explained before this finishes the proof.

The second subcase is that D_1 does not locate on the same connected component Γ_0 passing through C_1 . Then b must not be contained in the area bounded by Γ_0 by our construction and assumption on D_1 . Going along Γ_0 counter-clockwise, \arg_L will decrease 2π while \arg_R return to the same initial value. This shows that the difference of arguments δ is exactly 2π , which also finishes our proof by the same reason.

If a is on the lower half plane and b is on the upper half plane, we may choose the points C_2, D_2 instead, and the proof is the same.

When $a, b \in \mathbb{R}$ and one of them is positive, the triangle $\triangle Oab$ degenerates. Yet our proof is still valid without any essential modification. \square \square

Remark 6.2. Before finding the traditional proof to Lemma 6.1 as given above, we seek help from symbolic and numerical computations. Our colleague Professor Bican Xia, using an algorithm developed by him [12], succeeded in verifying the conclusion of Lemma 6.1. This was very important to make us believe the conclusions of Lemma 6.1 and Theorem 3.5, and to motivate us to find the proof given above. Professor Xia's method has been utilized in Maple (version 13 and later).

Compared to the previous situation where one obtains existence result, if the parameters are subject to different restrictions, one can prove non-existence result as below. This is used in Section 3 to verify the regularity of Example 3.6.

Lemma 6.3. *In Lemma 6.1, if we assume $a = b \in \mathbb{R}$ and $m = 1$, but drop the requirement of $a + b = -1$, then equation (37) has no solutions z when $-a$ is a sufficiently large positive real number (e.g. $-a > 1$)*

Proof. Under our assumptions, (37) simplifies to

$$|z - a|^2 = z^3/|z|^2. \quad (39)$$

So $z = r\omega^j$ for some $j \in \{0, 1, 2\}$ and $r > 0, \omega = e^{2\pi i/3}$. That means on the complex plane \mathbb{C} , the solution z , if it exists, must be located on the union of three radial lines. So we need only to compare $|z - a|^2$ and $|z|$, the modules at either sides of (39) for z in this subset. When $r > 0$ is small enough or big enough, the module $|z - a|^2$ is obviously larger than $|z|$. Thus intuitively we know that for suitable a there will always be $|z - a|^2 > |z|$. It is easy to rigorously verify this assertion; see the elementary and standard proof in [5] (the end of Section 7). This shows the non-existence of solution z . \square \square

7 Appendix B

The theorem below is the key lemma in our proof to Theorem 4.7 which shows the non-existence of a complete, non-oriented, algebraic stationary surface in \mathbb{R}_1^4 with total Gaussian curvature $2\pi(g+2)$ and without any singular points or singular ends.

Theorem 7.1. *Let \widetilde{M} be a closed oriented surface of genus g , $I : \widetilde{M} \rightarrow \widetilde{M}$ be an orientation-reversing involution of \widetilde{M} without fixed points. $\tilde{\phi} : \widetilde{M} \rightarrow S^2 \subset \mathbb{R}^3$ is an odd map, i.e., $\tilde{\phi}(I(p)) = -\tilde{\phi}(p)$. Then $\deg \tilde{\phi} \equiv g - 1 \pmod{2}$*

The statement reminds us of the famous theorem that any odd map $f : S^n \rightarrow S^n$ has odd degree, which implies the Borsuk-Ulam Theorem. We believe that this generalization is not a new result. Yet to the best of our knowledge we could not find a reference. The proof below is provided by Professor Fan Ding from Peking University.

Proof. Let $M = \widetilde{M}/\{p \sim I(p)\}$ be the quotient surface which is non-orientable. $\tilde{\phi} : \widetilde{M} \rightarrow S^2$ induces a quotient map ϕ from M to the projective plane $\mathbb{R}P^2 = S^2/\{x \sim -x\}$.

Decompose M as the connected sum of $g+1$ projective planes $M = M_1 \sharp \cdots \sharp M_{g+1}$. For any $M_j (j = 1, \dots, g+1)$, we choose a closed path γ_j in M_j representing the generator of the first homology group $H_1(M_j, \mathbb{Z}_2)$, which lifts to a path $\tilde{\gamma}_j \subset \widetilde{M}$ whose end points are a pair of antipodal points. As an odd map, $\tilde{\phi}$ maps $\tilde{\gamma}_j$ to another path connecting antipodal points, which projects to a closed path representing the generator of $H_1(\mathbb{R}P^2, \mathbb{Z}_2)$. Thus the induced map

$$\phi^* : H^1(\mathbb{R}P^2; \mathbb{Z}_2) \rightarrow H^1(M; \mathbb{Z}_2)$$

on the cohomology groups is given by

$$\phi^*(\alpha) = \alpha_1 + \cdots + \alpha_{g+1},$$

where $0 \neq \alpha \in H^1(\mathbb{R}P^2; \mathbb{Z}_2)$, and $\alpha_j \in H^1(M; \mathbb{Z}_2) (j = 1, \dots, g+1)$ satisfies $\alpha_j([\gamma_i]) = 0$ for $i \neq j$ and $\alpha_j([\gamma_j]) = 1$. Since the intersection between the homology classes $[\gamma_i] \in H_1(M; \mathbb{Z}_2)$ and $[\gamma_j] \in H_1(M; \mathbb{Z}_2)$ is 1 when $i = j$ and 0 when $i \neq j$, the Poincaré dual of α_j is $[\gamma_j]$. Thus the Poincaré dual of $\phi^*(\alpha)$ is $[\gamma_1] + \cdots + [\gamma_{g+1}]$. Since the self-intersection of the homology class $[\gamma_1] + \cdots + [\gamma_{g+1}] \in H_1(M; \mathbb{Z}_2)$ is $g+1 \pmod{2}$,

$$\phi^*(\alpha \cup \alpha) = \phi^*(\alpha) \cup \phi^*(\alpha) = (g+1)\beta,$$

where $0 \neq \beta \in H^2(M; \mathbb{Z}_2)$. Hence the mod 2 degree of ϕ is $g+1 \pmod{2}$. Thus the mod 2 degree of $\tilde{\phi}$ is $g+1 \pmod{2}$. This finishes the proof. \square \square

Remark 7.2. If we only consider a continuous map ϕ from the non-oriented quotient surface M to $\mathbb{R}P^2$, then the conclusion is not necessarily true. The simplest counter-example is a constant map. On the other hand, if we assume that ϕ is a branched covering map, then the conclusion is one part of Meeks' Theorem 1 in [8]. We don't know whether our conclusion could be generalized to the case of odd mapping $\tilde{\phi} : \tilde{M}_1 \rightarrow \tilde{M}_2$ where each closed oriented surface is endowed with an orientation-reversing involution without fixed points.

References

- [1] Alías, L. J., Palmer, B. *Curvature properties of zero mean curvature surfaces in four-dimensional Lorentzian space forms*, Math. Proc. Camb. Phil. Soc. 124, 315-327 (1998)
- [2] Costa, C. J. *Classification of complete minimal surfaces in R^3 with total curvature 12π* , Invent. math. 105, 273-303 (1991)
- [3] Huber, A. *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. 32, 13-72 (1957)
- [4] Jorge, L. P. M., Meeks, III W. H. *The topology of complete minimal surfaces of finite total gaussian curvature*, Topology 22, 203-221 (1983)
- [5] Liu, Z., Ma, X., Wang, C., Wang, P., *Global geometry and topology of spacelike stationary surfaces in \mathbb{R}_1^4* , arxiv:1103.4700v5[math.DG]
- [6] López, F. *The classification of complete minimal surfaces with total curvature greater than -12π* , Trans. Amer. Math. Soc. 334, 49-74 (1992)
- [7] Martin, F. *Complete nonorientable minimal surfaces in \mathbb{R}^3* , In: Clay Mathematics Proceedings, vol. 2, pp. 371-380 (2005)
- [8] Meeks, III W. H. *The classification of complete minimal surfaces in R^3 with total curvature greater than -8π* , Duke Math. J. 48, no. 3, 523-535 (1981)
- [9] de Oliveira, M. Elisa *Some new examples of nonoriented minimal surfaces*, Proc. Amer. Math. Soc. 98, 629-636 (1986)
- [10] Osserman, R. *A survey of minimal surfaces*, Second edition. Dover Publications, Inc., New York, 1986
- [11] Schoen, R. *Uniqueness, symmetry and embeddedness of minimal surfaces*, J. Diff. Geom. 18, 791-809 (1983)
- [12] Yang, L., Xia, B. *Real solution classifications of a class of parametric semi-algebraic systems*. In: Algorithmic Algebra and Logic — Proceedings of the A3L 2005 (A. Dolzmann, A. Seidl, and T. Sturm, eds.), pp. 281-289. Herstellung und Verlag, Norderstedt (2005)

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